Tutorial 6

March 3, 2016

1. Discuss the graphs of the eigenfunction $X_n(x) = \sin(\frac{n\pi x}{l})$ for $n = 1, 2, 3, 4$.

   Solution: See figure 1 on page 84 or the following figure: The red line represents $\sin(\frac{\pi x}{l})$, the purple line represents $\sin(\frac{2\pi x}{l})$, the orange line represents $\sin(\frac{3\pi x}{l})$ and the black line represents $\sin(\frac{4\pi x}{l})$.

   Note that the minimal eigenvalue is $(\frac{\pi}{l})^2$ which is called the principal eigenvalue, and its corresponding eigenfunction is $\sin(\frac{\pi x}{l})$ which is always positive when $0 < x < l$.

2. Using the method of separation of variables to solve the problem:

$$\begin{cases} u_t - ku_{xx} = 0, 0 < x < l, t > 0 \\ u_x(0,t) = 0, u(l,t) = 0, \\ u(x, t = 0) = \phi(x) \end{cases}$$

   Solution: Let $u(x,t) = T(t)X(x)$, we have

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

   Actually, λ is positive. Therefore, $T(t)$ satisfies the equation $T' = -\lambda kT$, whose solution is $T(t) = Ae^{-\lambda kt}$. Furthermore,

$$-X'' = \lambda X, X'(0) = X(l) = 0.$$

   So by solving the above DE, the eigenvalues are $[\frac{(n+\frac{1}{2})\pi}{l}]^2$, the eigenfunctions are $X_n(x) = \cos\frac{(n+\frac{1}{2})\pi x}{l}$ for $n = 0, 1, 2, \cdots$, and the solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-[\frac{(n+\frac{1}{2})\pi}{l}]^2 kt} \cos\frac{(n+\frac{1}{2})\pi x}{l}.$$

   provided that

$$\phi(x) = \sum_{n=0}^{\infty} A_n \cos\frac{(n+\frac{1}{2})\pi x}{l}.$$

3. Using the method at the end of Page 86 to show that all the eigenvalues for

$$-X''(x) = \lambda X(x)$$

$$X(0) = X(l) = 0$$

   are positive.

   Solution: Case 1: If $\lambda = 0$, then $X''(x) = 0$. The general solution is

$$X(x) = ax + b$$

   where a, b are constants. And $X(0) = X(l) = 0$ implies that $a = b = 0$, so that $X(x) = 0$. Therefore 0 is not an eigenvalue.

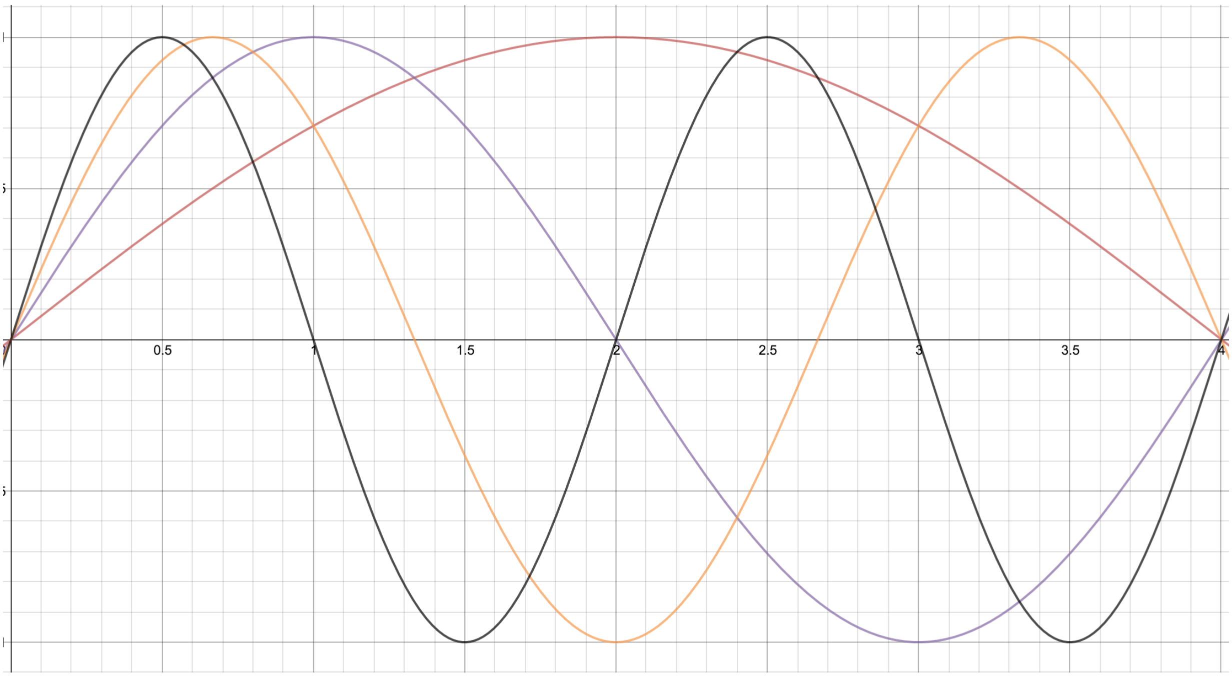

Figure 1: The graphs of eigenfunctions

Case 2: If $\lambda < 0$, there exists $\gamma > 0$ such that $\lambda = -\gamma^2$. Then $X''(x) - \gamma^2 X(x) = 0$. The general solution is

$$X(x) = Ae^{\gamma x} + Be^{-\gamma x}$$

where A, B are constants. And $X(0) = X(l) = 0$ implies that $A = B = 0$, so that $X(x) = 0$. Therefore λ can not be positive.

Case 3: Let λ be complex number. Let γ be either one of the two square roots of $-\lambda$, the other one is $-\gamma$. Then the general solution of $X''(x) + \lambda X(x) = 0$ is

$$X(x) = Ce^{\gamma x} + De^{-\gamma x}$$

where we are using complex exponential function. The boundary conditions yeild

$$0 = X(0) = C + D$$

$$0 = Ce^{\gamma l} + De^{-\gamma l}$$

Therefore $e^{2\gamma l} = 1$ which implies that $Re(\gamma) = 0$ and $2lIm(\gamma) = 2\pi n$ for $n = 1, 2, \ldots$ Hence $\gamma = n\pi i/l$ and $\lambda = -\gamma^2 = n^2\pi^2/l^2$, which is real and positive.Thus the only eigenvalues λ are positive numbers; in fact, they are $(n\pi/l)^2$, $n = 1, 2, \ldots$.